

Boundary value problems associated to a Hermitian Helmholtz equation

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Abstract

As is the case for the Laplace operator, in Euclidean Clifford analysis also the Helmholtz operator can be factorized, more precisely by using perturbed Dirac operators. In this paper we consider the Helmholtz equation in a circulant matrix form in the context of Hermitian Clifford analysis. The aim is to introduce and study the corresponding inhomogeneous Hermitian Dirac operators, which will constitute a splitting of the traditional perturbed Dirac operators of the Euclidean Clifford analysis context. This will not only lead to special solutions of the Hermitian Helmholtz equation as such, but also to the study of boundary value problems of Riemann type for those solutions, which are, in fact, solutions of the Hermitian perturbed Dirac operators involved.

Keywords. Hermitian Clifford analysis, Helmholtz equations, boundary value problems.

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1 Introduction

If the Sturm-Liouville equation can be seen as the most important differential equation in one dimension, then the Helmholtz equation $\Delta\psi + k^2\psi$ may well deserve that title in higher dimension. For $k^2 > 0$ it usually arises as the space part of the wave equation, where k is then called the wave number, and because of this intimate connection to the wave equation, it arises in the context of applications such as electromagnetic radiation, seismology, acoustics and also quantum mechanics. Moreover, for $k^2 < 0$, so when k is imaginary, the corresponding Helmholtz equation may be obtained as the space part of the diffusion equation.

In so-called Euclidean Clifford analysis, a function theory on Euclidean space \mathbb{R}^m which is to be regarded as a generalization to higher dimensions of the theory

of holomorphic functions in the complex plane, the Helmholtz operator has been factorized by means of so-called perturbed Dirac operators, also called k -Dirac operators. Since, in particular cases, i.e. with specific assumptions on the boundary data, the boundary value problem for such a k -Dirac operator reduces to a Maxwell system, see e.g. [28], there has been a great deal of interest in the corresponding function theory, both from a mathematics and from a physics point of view. For details, we refer the reader to [20, 26, 27, 24, 34, 35, 36, 37, 8, 19]. In particular, in the physically relevant lower dimensional context, techniques from quaternionic analysis have been applied, see the books [21, 22, 23] and the references therein.

More recently Hermitian Clifford analysis has emerged as a refinement of the Euclidean Clifford framework for the case of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. One possible way for introducing it, is to equip the vector space \mathbb{R}^{2n} with a (almost) complex structure, i.e. an $\text{SO}(2n; \mathbb{R})$ element J for which $J^2 = -\mathbf{1}$. In fact, it is precisely in order to ensure that such a complex structure exists, that the dimension of the underlying vector space has to be taken even. Here, Hermitian monogenic functions are considered, i.e. functions taking values either in the complex Clifford algebra \mathbb{C}_{2n} or in complex spinor space, which are simultaneous null solutions of two complex Hermitian Dirac operators, constituting a splitting of the traditional Dirac operator. The resulting function theory may thus be seen as a refinement of Euclidean Clifford analysis. The study of complexified Dirac operators (also in other settings) was initiated in [31, 29, 32]; a systematic development of Hermitian Clifford analysis is still in full progress, see e.g. [17, 9, 10, 13, 16, 14, 33, 7]. In the course of these studies, it turned out that a matrix approach, using circulant (2×2) matrix functions, was the key to obtain some corner stone results, such as Cauchy and Borel–Pompeiu integral formulae. For details on this approach, we refer to [3, 4, 5, 6, 15, 11, 12].

In this paper, we will consider the Helmholtz equation in a (2×2) circulant matrix form, in the framework of Hermitian Clifford analysis, and we will establish a decomposition of the matrix Helmholtz operator by means of inhomogeneous, or perturbed, Hermitian Dirac operators, which constitute a splitting of the perturbed Dirac operators used in the Euclidean Clifford setting. This will lead to a study of boundary value problems of Riemann type for the corresponding matrix operators, which may be seen as refinements of the boundary value problems mentioned above and thus also of the resulting Maxwell systems.

The outline of the paper is as follows. For the convenience of the reader, we recall in Section 2 some basic concepts and results of the theory of Hermitian monogenic functions, both in the scalar and in the matricial context. In Section 3 we consider the Helmholtz equation in the Hermitian Clifford context and we factorize it by means of inhomogeneous, or perturbed, Hermitian Dirac operators, for which we set up the function theory in Section 4. And finally, in Section 5, we deal with

boundary value problems of Riemann type for these operators. In this section, we first let the operators act on spaces of Hölder continuous matrix functions, but the same results can also be obtained by similar reasoning for Lebesgue p -integrable matrix functions, as we will argue in a number of remarks concerning that case.

2 The Hermitian Clifford analysis setting

Let (e_1, \dots, e_m) be an orthonormal basis of Euclidean space \mathbb{R}^m and consider the complex Clifford algebra \mathbb{C}_m constructed over \mathbb{R}^m . The non-commutative multiplication in \mathbb{C}_m is governed by the rules:

$$e_j^2 = -1, \quad j = 1, \dots, m, \quad e_j e_k + e_k e_j = 0, \quad j, k = 1, \dots, m, \quad j \neq k$$

Then, \mathbb{C}_m is generated additively by elements of the form $e_A = e_{j_1} \dots e_{j_k}$, where $A = \{j_1, \dots, j_k\} \subset \{1, \dots, m\}$ with $j_1 < \dots < j_k$, while for $A = \emptyset$, one puts $e_\emptyset = 1$, the identity element. Any Clifford number $\lambda \in \mathbb{C}_m$ may thus be written as $\lambda = \sum_A \lambda_A e_A$, $\lambda_A \in \mathbb{C}$, its Hermitian conjugate λ^\dagger being defined by $\lambda^\dagger = \sum_A \lambda_A^c \bar{e}_A$, where the bar denotes the real Clifford algebra conjugation, i.e. the main anti-involution for which $\bar{e}_j = -e_j$, and λ_A^c stands for the complex conjugate of the complex number λ_A . Euclidean space \mathbb{R}^m is embedded in the Clifford algebra \mathbb{C}_m by identifying (x_1, \dots, x_m) with the real Clifford vector \underline{X} given by $\underline{X} = \sum_{j=1}^m e_j x_j$, for which $\underline{X}^2 = -\langle \underline{X}, \underline{X} \rangle = -|\underline{X}|^2$. The Fischer dual of \underline{X} is the vector valued first order Dirac operator $\partial_{\underline{X}} = \sum_{j=1}^m e_j \partial_{x_j}$, factorizing the Laplacian: $\Delta_m = -\partial_{\underline{X}}^2$; it underlies the notion of monogenicity of a function, the higher dimensional counterpart of holomorphy in the complex plane. The considered functions are defined on (open subsets of) \mathbb{R}^m and take values in the Clifford algebra \mathbb{C}_m . They are of the form $g = \sum_A g_A e_A$, with g_A complex valued. Whenever a property such as continuity, differentiability, etc. is ascribed to g it is meant that all components g_A show that property. A Clifford algebra valued function g , defined and differentiable in an open region Ω of \mathbb{R}^m , is then called (left) monogenic in Ω iff $\partial_{\underline{X}} g = 0$ in Ω .

The transition from Euclidean Clifford analysis to the Hermitian Clifford setting is essentially based on the introduction of a complex structure J , also referred to as an almost complex structure, i.e. a particular $\text{SO}(m)$ element, satisfying $J^2 = -\mathbf{1}_m$. Since such an element can only exist when the dimension of the vector space is even, we put $m = 2n$ from now on. In terms of the orthonormal basis, a particular realization of the complex structure is $J[e_{2j-1}] = -e_{2j}$ and $J[e_{2j}] = e_{2j-1}$, $j = 1, \dots, n$. Two projection operators $\pm \frac{1}{2}(\mathbf{1}_{2n} \pm iJ)$, associated to J , then produce the main objects of Hermitian Clifford analysis by acting upon the corresponding objects in the Euclidean setting, see [9, 10]. The real Clifford vector and its corresponding Dirac operator are now denoted

$$\underline{X} = \sum_{j=1}^n (e_{2j-1} x_{2j-1} + e_{2j} x_{2j}), \quad \partial_{\underline{X}} = \sum_{j=1}^n (e_{2j-1} \partial_{x_{2j-1}} + e_{2j} \partial_{x_{2j}})$$

while we will also consider their so-called 'twisted' counterparts, obtained through the action of J , i.e.

$$\underline{X}| = \sum_{j=1}^n (e_{2j-1}x_{2j} - e_{2j}x_{2j-1}), \quad \partial_{\underline{X}|} = \sum_{j=1}^n (e_{2j-1}\partial_{x_{2j}} - e_{2j}\partial_{x_{2j-1}})$$

The projections of the vector variable \underline{X} then yield the Hermitian Clifford variables \underline{Z} and \underline{Z}^\dagger , given by

$$\underline{Z} = \frac{1}{2}(\underline{X} + i\underline{X}|) \quad \text{and} \quad \underline{Z}^\dagger = -\frac{1}{2}(\underline{X} - i\underline{X}|)$$

and those of the Dirac operator $\partial_{\underline{X}}$ yield (up to a constant factor) the Hermitian Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$, given by

$$\partial_{\underline{Z}^\dagger} = \frac{1}{4}(\partial_{\underline{X}} + i\partial_{\underline{X}|}) \quad \text{and} \quad \partial_{\underline{Z}} = -\frac{1}{4}(\partial_{\underline{X}} - i\partial_{\underline{X}|})$$

The Hermitian vector variables and Dirac operators are isotropic, i.e. $(\underline{Z})^2 = (\underline{Z}^\dagger)^2 = 0$ and $(\partial_{\underline{Z}})^2 = (\partial_{\underline{Z}^\dagger})^2 = 0$, whence the Laplacian allows for the decomposition $\Delta_{2n} = 4(\partial_{\underline{Z}}\partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}^\dagger}\partial_{\underline{Z}})$. These objects lie at the core of the Hermitian function theory by means of the following definition (see e.g. [9, 17]).

Definition 1 *A continuously differentiable function g in $\Omega \subset \mathbb{R}^{2n}$ with values in \mathbb{C}_{2n} is called (left) h -monogenic in Ω , iff it satisfies in Ω the system $\partial_{\underline{Z}}g = 0 = \partial_{\underline{Z}^\dagger}g$ or, equivalently, the system $\partial_{\underline{X}}g = 0 = \partial_{\underline{X}|}g$.*

Hermitian monogenicity thus constitutes a refinement of monogenicity, since h -monogenic functions are monogenic w.r.t. both Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$.

The respective fundamental solutions of $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$, i.e. the Cauchy kernels for the corresponding theories, are

$$E(\underline{X}) = -\frac{1}{\sigma_{2n}} \frac{\underline{X}}{|\underline{X}|^{2n}}, \quad E|(\underline{X}) = -\frac{1}{\sigma_{2n}} \frac{\underline{X}|}{|\underline{X}|^{2n}}, \quad \underline{X} \in \mathbb{R}^{2n} \setminus \{0\}$$

Here σ_{2n} is the surface area of the unit sphere in \mathbb{R}^{2n} . The transition from Hermitian Clifford analysis to a circulant matrix approach is essentially based on the following observation. Let $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$ be the circulant (2×2) -matrix Dirac operator given by

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} = \begin{pmatrix} \partial_{\underline{Z}} & \partial_{\underline{Z}^\dagger} \\ \partial_{\underline{Z}^\dagger} & \partial_{\underline{Z}} \end{pmatrix}$$

and consider, see [29], the matrix

$$\mathcal{E} = \begin{pmatrix} \mathcal{E} & \mathcal{E}^\dagger \\ \mathcal{E}^\dagger & \mathcal{E} \end{pmatrix}$$

with $\mathcal{E} = -(E + iE|)$ and $\mathcal{E}^\dagger = (E - iE|)$. Then $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathcal{E} = \delta$, where δ is the diagonal matrix with the Dirac delta distribution δ on the diagonal, whence \mathcal{E} may be considered as a fundamental solution of the matrix operator $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$. This has been the first step towards important results such as the Borel–Pompeiu and Cauchy integral representation formulae and the Teodorescu operator, see below.

Moreover, this has also lead to a theory of **H**–monogenic (2×2) circulant matrix functions, the framework for this theory being as follows. Let g_1, g_2 be continuously differentiable functions defined in Ω and taking values in \mathbb{C}_{2n} , and consider the corresponding (2×2) circulant matrix function

$$\mathbf{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}$$

The ring of such matrix functions over \mathbb{C}_{2n} is denoted by $\mathbf{CM}^{2 \times 2}$. In what follows, \mathbf{O} will denote the matrix in $\mathbf{CM}^{2 \times 2}$ with zero entries.

Definition 2 *The matrix function $\mathbf{G}_2^1 \in \mathbf{CM}^{2 \times 2}$ is called (left) **H**–monogenic in Ω if and only if it satisfies in Ω the system $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} [\mathbf{G}_2^1] = \mathbf{O}$.*

The space of **H**–monogenic functions on Ω is denoted $\mathbf{HM}(\Omega)$. In general, the **H**–monogenicity of \mathbf{G}_2^1 does not imply the h –monogenicity of its entries g_1 and g_2 . However, choosing $g_1 = g$ and $g_2 = 0$, the **H**–monogenicity of the corresponding diagonal matrix \mathbf{G}_0 is seen to be equivalent to the h –monogenicity of the function g .

Notions of continuity, differentiability and integrability of $\mathbf{G}_2^1 \in \mathbf{CM}^{2 \times 2}$ have the usual component–wise meaning. In particular, we will need to define in this way the classes $\mathbf{C}^s(\mathbf{E})$, $s \in \mathbb{N} \cup \{0\}$, of s times continuously differentiable functions over some suitable subset \mathbf{E} of \mathbb{R}^{2n} , as well as the classes $\mathbf{L}_p(\mathbf{E})$ and $\mathbf{C}^{0, \nu}(\mathbf{E})$ of, respectively, Lebesgue p –integrable and Hölder continuous circulant matrix functions over \mathbf{E} . However, introducing the non–negative function

$$\|\mathbf{G}_2^1(\underline{X})\| = \max\{|g_1(\underline{X})|, |g_2(\underline{X})|\}$$

the latter classes may also be defined by means of the traditional conditions

$$\|\mathbf{G}_2^1\|_p := \left(\int_{\mathbf{E}} \|\mathbf{G}_2^1(\underline{X})\|^p \right)^{\frac{1}{p}} < +\infty,$$

and

$$|\mathbf{G}_2^1|_{\nu, \mathbf{E}} := \sup_{\underline{X}, \underline{Y} \in \mathbf{E}; \underline{X} \neq \underline{Y}} \frac{\|\mathbf{G}_2^1(\underline{X}) - \mathbf{G}_2^1(\underline{Y})\|}{|\underline{X} - \underline{Y}|^\nu} < +\infty$$

respectively, where

$$\|\mathbf{G}_2^1\|_{\nu, \mathbf{E}} := \max_{\underline{X} \in \mathbf{E}} \|\mathbf{G}_2^1(\underline{X})\| + |\mathbf{G}_2^1|_{\nu, \mathbf{E}}$$

is the norm of the element $\mathbf{G}_2^1 \in \mathbf{C}^{0, \nu}(\mathbf{E})$.

Moreover, we say that a matrix function from $\mathbf{CM}^{2 \times 2}$ exhibits a certain behaviour (e.g. weakly singular or the like) if all its entries show that behaviour.

We will now recall some integral representation formulae and derived results for \mathbf{H} -monogenic functions. For simplicity, throughout the remainder of the paper, we will assume that Ω is a Jordan domain in \mathbb{R}^{2n} , and we put $\Omega^+ = \Omega$ and $\Omega^- = \mathbb{R}^{2n} \setminus \overline{\Omega^+}$, where both open sets are assumed to be connected. Furthermore, we assume the boundary Γ of Ω to be a $(2n-1)$ -dimensional compact topological and oriented hypersurface of standard class, e.g. piecewise C^∞ smooth, Liapunov or Lipschitz. Although we may conjecture that the results will not depend on this particular geometric assumption, we have not yet developed the arguments for other geometries.

The notations \underline{Y} and $|\underline{Y}|$ will be reserved for Clifford vectors associated to points in Ω^\pm , while their Hermitian counterparts are denoted $\underline{V} = \frac{1}{2}(\underline{Y} + i|\underline{Y}|)$ and $\underline{V}^\dagger = -\frac{1}{2}(\underline{Y} - i|\underline{Y}|)$. By means of the matrix approach sketched above, the following Hermitian Borel–Pompeiu formula was established in [12].

Theorem 1 *Let $\mathbf{G}_2^1 \in C^1(\overline{\Omega})$. It then holds that*

$$\mathcal{C}_\Gamma[\mathbf{G}_2^1](\underline{Y}) + \mathcal{T}_\Omega[\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{Y})] = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

where $\mathcal{C}_\Gamma \mathbf{G}_2^1$ is the Hermitian Cauchy integral given by

$$\mathcal{C}_\Gamma[\mathbf{G}_2^1](\underline{Y}) = \int_\Gamma \mathcal{E}(\underline{Z} - \underline{V}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{X}) d\mathcal{H}^{2n-1}, \quad \underline{Y} \in \Omega^\pm \quad (1)$$

the circulant matrix $\mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)}$ containing (up to a constant factor) the Hermitian projections \underline{N} and $-\underline{N}^\dagger$ of the unit normal vector $\underline{n}(\underline{X})$ at the point $\underline{X} \in \Gamma$. Furthermore

$$d\mathcal{H}^{2n-1} = \begin{pmatrix} d\mathcal{H}^{2n-1} & 0 \\ 0 & d\mathcal{H}^{2n-1} \end{pmatrix}$$

where \mathcal{H}^{2n-1} denotes the $(2n-1)$ -dimensional Hausdorff measure, and finally, \mathcal{T}_Ω denotes the Hermitian Teodorescu transform, given for $\mathbf{F}_2^1 \in C^1(\Omega)$ by

$$\mathcal{T}_\Omega[\mathbf{F}_2^1(\underline{Y})] = - \int_\Omega \mathcal{E}(\underline{Z} - \underline{V}) \mathbf{F}_2^1(\underline{X}) dW(\underline{Z}, \underline{Z}^\dagger), \quad \underline{Y} \in \mathbb{R}^{2n}$$

where $dW(\underline{Z}, \underline{Z}^\dagger)$ is the associated volume element defined through

$$dV(\underline{X}) = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n dW(\underline{Z}, \underline{Z}^\dagger)$$

Corollary 1 *[Cauchy formula] Let $\mathbf{G}_2^1 \in \mathbf{C}^1(\overline{\Omega}) \cap \mathbf{HM}(\Omega)$. It then holds that*

$$\mathcal{C}_\Gamma[\mathbf{G}_2^1](\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

The following theorem expresses the basic property of the matricial Hermitian Teodorescu transform to be the algebraic right inverse to the operator $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$. This result can be found in [4].

Theorem 2 *If $\mathbf{G}_2^1 \in \mathbf{C}^1(\Omega) \cap \mathbf{C}(\overline{\Omega})$, then*

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathcal{T}_\Omega[\mathbf{G}_2^1](\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

Based on the above definition (1) of the Hermitian Cauchy integral, which is in fact defined for any $\mathbf{G}_2^1 \in \mathbf{C}^{0,\nu}(\Gamma)$, a matricial Hermitian Hilbert transform was introduced in [2], by considering boundary limits which take the form of the usual Plemelj–Sokhotski formulae.

Theorem 3 *Let $\mathbf{G}_2^1 \in \mathbf{C}^{0,\nu}(\Gamma)$. Then the boundary values of the Hermitian Cauchy integral $\mathcal{C}_\Gamma \mathbf{G}_2^1$ are given by*

$$\lim_{\substack{\underline{Y} \rightarrow \underline{U} \\ \underline{Y} \in \Omega^\pm}} \mathcal{C}_\Gamma[\mathbf{G}_2^1](\underline{Y}) = (-1)^{\frac{n(n+1)}{2}} (2i)^n \left(\pm \frac{1}{2} \mathbf{G}_2^1(\underline{U}) + \frac{1}{2} \mathcal{H}_\Gamma[\mathbf{G}_2^1](\underline{U}) \right), \quad \underline{U} \in \Gamma$$

where \mathcal{H}_Γ is the Hermitian Hilbert transform defined as

$$\mathcal{H}_\Gamma[\mathbf{G}_2^1](\underline{U}) = 2 \int_{\Gamma} \mathcal{E}(\underline{Z} - \underline{W}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{X}) d\mathcal{H}^{2n-1}, \quad \underline{U} \in \Gamma$$

Moreover, the basic properties of a Hilbert transform hold, see [2].

Theorem 4 *The Hermitian Hilbert transform \mathcal{H}_Γ shows the following properties*

- \mathcal{H}_Γ is a bounded operator on $\mathbf{C}^{0,\nu}(\Gamma)$, i.e., there exist a real constant c_Γ such that for any $\mathbf{G}_2^1 \in \mathbf{C}^{0,\nu}(\Gamma)$

$$\|\mathcal{H}_\Gamma[\mathbf{G}_2^1]\|_{\nu, \Gamma} \leq c_\Gamma \|\mathbf{G}_2^1\|_{\nu, \Gamma}.$$

- \mathcal{H}_Γ is an involution on $\mathbf{C}^{0,\nu}(\Gamma)$, that is, $\mathcal{H}_\Gamma^2 = \mathbf{1}_0$, where $\mathbf{1}_0$ denotes the identity in $\mathbf{CM}^{2 \times 2}$.

Here c_Γ denotes a generic constant depending only on Γ .

Remark 1 *In Theorems 3 and 4, we are assuming that $\mathbf{G}_2^1 \in \mathbf{C}^{0,\nu}(\Gamma)$ and hence all integrals are understood in the Riemann sense (proper or improper). If now $\mathbf{G}_2^1 \in \mathbf{L}_p(\Gamma)$ then one has to understand $\mathcal{C}_\Gamma \mathbf{G}_2^1$ as a Lebesgue integral, and the necessary changes can be easily made. For example, the limits in Theorem 3 exist almost everywhere on Γ with respect to the surface Lebesgue measure. An \mathbf{L}_p formulation of Theorem 4 follows from standard Calderon–Zygmund theory and recalling that $\mathbf{C}^{0,\nu}(\Gamma)$ is dense in $\mathbf{L}_p(\Gamma)$ by classical arguments.*

3 The matricial Helmholtz equation

As mentioned in the introduction the Helmholtz equation for a \mathbb{C}_{2n} -valued function g reads

$$\Delta_{2n}g + k^2g = 0 \quad (2)$$

where k is a given real constant. Then introduce the complex constants

$$K = -\frac{1}{4}(k - i k), \quad K^\dagger = -\frac{1}{4}(k + i k) \quad (3)$$

and observe that

$$\begin{aligned} \Delta_{2n} + k^2 &= 4[(\partial_{\underline{Z}} + K)(\partial_{\underline{Z}^\dagger} + K^\dagger) + (\partial_{\underline{Z}^\dagger} - K^\dagger)(\partial_{\underline{Z}} - K)] \\ &= -(\partial_{\underline{X}} + k)(\partial_{\underline{X}} - k) = -(\partial_{\underline{X}|} + k)(\partial_{\underline{X}|} - k) \end{aligned}$$

The Helmholtz equation (2) may thus be written as

$$[(\partial_{\underline{Z}} + K)(\partial_{\underline{Z}^\dagger} + K^\dagger) + (\partial_{\underline{Z}^\dagger} - K^\dagger)(\partial_{\underline{Z}} - K)]g = 0,$$

or, equivalently, as

$$(\partial_{\underline{X}} + k)(\partial_{\underline{X}} - k)g = 0 \quad \text{or} \quad (\partial_{\underline{X}|} + k)(\partial_{\underline{X}|} - k)g = 0$$

Now, let Δ be the matricial Laplacian, i.e.

$$\Delta = \begin{pmatrix} \Delta_{2n} & 0 \\ 0 & \Delta_{2n} \end{pmatrix}.$$

Observe that

$$4\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}(\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)})^\dagger = 4(\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)})^\dagger \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} = \Delta$$

whence the Hermitian Dirac matrix operator may be said to factorize the Laplacian in some sense. The above factorization also means that \mathbf{H} -monogenic matrix functions of class $\mathbf{C}^2(\Omega)$ are harmonic in Ω , meaning that they belong to $\ker(\Delta)$.

We may also consider the matricial Helmholtz operator $\Delta + k^2\mathbf{1}_0$, a formal factorization of which is then given by

$$\Delta + k^2\mathbf{1}_0 = 4(\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} + \mathcal{K})(\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)})^\dagger + \mathcal{K}^\dagger \quad (4)$$

$$= 4((\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)})^\dagger + \mathcal{K}^\dagger)(\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} + \mathcal{K}) \quad (5)$$

where

$$\mathcal{K} = \begin{pmatrix} K & -K^\dagger \\ -K^\dagger & K \end{pmatrix}, \quad \mathcal{K}^\dagger = \begin{pmatrix} K^\dagger & -K \\ -K & K^\dagger \end{pmatrix} \quad (6)$$

Seen the factorization (4)–(5), the null solutions of the operator

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^\mathcal{K} = \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} + \mathcal{K}$$

called the perturbed Hermitian Dirac matrix operator, are special solutions of the Hermitian Helmholtz equation, whence we will set up a function theory associated to this operator. We will call its null solutions \mathcal{K} -Hermitian monogenic functions, and denote $\mathbf{HM}^{\mathcal{K}}(\Omega) = \ker \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^{\mathcal{K}}$. The corresponding systems for the components of the considered circulant matrix functions will be given in the next section.

4 Function theory for $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^{\mathcal{K}}$

Following the ideas above, we will now develop a function theory associated to the perturbed Hermitian Dirac matrix operator. As a first important step in this development, it is necessary to construct its fundamental solution. First, note that, for $k \in \mathbb{R}$, a fundamental solution for the Helmholtz operator $\Delta_{2n} + k^2$ in \mathbb{R}^{2n} is given by

$$\theta_k(\underline{X}) = \left(-\frac{1}{2\pi r} \frac{\partial}{\partial r} \right)^{n-1} \left(-\frac{i}{4} H_0^{(1)}(kr) \right)$$

where $r = |\underline{X}|$ and $H_0^{(1)}$ is the usual zero-order Hankel function of the first kind. The main properties of this function can be found in e.g. [26, Proposition 3.1]. The complete proofs can be found in e.g. [18, p. 59–74]. For what follows it is important to mention that

$$\theta_k(\underline{X}) = \theta(\underline{X}) + O(|\underline{X}|^{-2n+4})$$

as $|\underline{X}| \rightarrow 0$, where $\theta(\underline{X})$ denotes the fundamental solution of the Laplacian, which itself behaves as $O(|\underline{X}|^{-2n+2})$ when $|\underline{X}| \rightarrow 0$.

The fundamental solutions of the perturbed Euclidean Dirac operators $\partial_{\underline{X}} + k$ and $\partial_{\underline{X}^\dagger} + k$ then respectively are

$$\begin{aligned} E_k(\underline{X}) &= -(\partial_{\underline{X}} - k) \theta_k(\underline{X}) \\ E|_k(\underline{X}) &= -(\partial_{\underline{X}^\dagger} - k) \theta_k(\underline{X}) \end{aligned}$$

which are known to show the following asymptotic behaviour for $|\underline{X}| \rightarrow 0$, see e.g. [26, p. 824]:

$$E_k(\underline{X}) = E(\underline{X}) + k\theta_k(\underline{X}) + O(|\underline{X}|^{-2n+3}) \quad (7)$$

$$E|_k(\underline{X}) = E|(\underline{X}) + k\theta_k(\underline{X}) + O(|\underline{X}|^{-2n+3}) \quad (8)$$

so that, still for $|\underline{X}| \rightarrow 0$, the differences $E_k(\underline{X}) - E(\underline{X})$ and $E|_k(\underline{X}) - E|(\underline{X})$ both behave as $O(|\underline{X}|^{-2n+2})$.

Moreover E_k and $E|_k$ are locally integrable in \mathbb{R}^{2n} and satisfy an appropriate decay condition at infinity, more precisely $O(|\underline{X}|^{-\frac{(2n-1)}{2}})$. For a full treatment of the behaviour at infinity of these functions we refer the reader to [24, Section 3].

Remark 2 *It is worth mentioning that, by Rellich's lemma, any solution g in the whole of \mathbb{R}^{2n} of the Helmholtz equation (2) with non-zero real k , which satisfies $g(\underline{X}) = O(|\underline{X}|^{-\frac{(2n-1)}{2}})$ in a (connected) neighbourhood of infinity in \mathbb{R}^{2n} , must vanish identically. For more details on this property, we refer to [24, Lemma 8.1].*

Starting from the pair of fundamental solutions $(E_k, E|_k)$ of the perturbed Euclidean Dirac operators $\partial_{\underline{X}} + k$ and $\partial_{\underline{X}|} + k$, we now construct the distributions

$$\mathcal{E}_K = -(E_k + iE|_k), \quad \mathcal{E}_K^* = (E_k - iE|_k)$$

Explicitly they are given by

$$\begin{aligned} \mathcal{E}_K(\underline{Z}) &= 4 \left[(\partial_{\underline{Z}^\dagger} + K^\dagger) \theta_k(\underline{Z}) \right] \\ \mathcal{E}_K^*(\underline{Z}) &= 4 \left[(\partial_{\underline{Z}} - K) \theta_k(\underline{Z}) \right] \end{aligned}$$

Observe that, when $k = 0$ and thus $K = 0$, we have that $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{E}_0^* = \mathcal{E}^\dagger$. Similarly to the Hermitian Dirac case, also here the Hermitian kernels \mathcal{E}_K and \mathcal{E}_K^* are not the fundamental solutions of the respective perturbed Hermitian Dirac operators $\partial_{\underline{Z}} + K$ and $\partial_{\underline{Z}^\dagger} - K^\dagger$. However, the (2×2) circulant matrix

$$\mathcal{E}_K = \begin{pmatrix} \mathcal{E}_K & \mathcal{E}_K^* \\ \mathcal{E}_K^* & \mathcal{E}_K \end{pmatrix} \quad (9)$$

can be seen as a fundamental solution of the operator $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^\mathcal{K}$, since we have that $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^\mathcal{K} \mathcal{E}_K = \delta$. On account of the asymptotic behaviour of the kernels $E_k(\underline{X})$ and $E|_k(\underline{X})$, also the matrix \mathcal{E}_K may be said to satisfy the decay condition $O(|\underline{X}|^{-\frac{(2n-1)}{2}})$ at infinity.

All of the above now inspires the following definition.

Definition 3 *Let g_1, g_2 be continuously differentiable functions defined in Ω and taking values in \mathbb{C}_{2n} , and consider the corresponding (2×2) circulant matrix function \mathbf{G}_2^1 . Then \mathbf{G}_2^1 is called (left) Helmholtz \mathbf{H} -monogenic in Ω if and only if it satisfies in Ω the system*

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^\mathcal{K} [\mathbf{G}_2^1] = \mathbf{0}.$$

Let us now explicitly write down the corresponding systems for the components of the considered circulant matrix functions, either in terms of the Hermitian Dirac operators, or in terms of the classical Dirac operator and its twisted version:

$$\begin{aligned} \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^\mathcal{K} [\mathbf{G}_2^1] = \mathbf{0} &\iff \begin{cases} (\partial_{\underline{Z}} + K)g_1 + (\partial_{\underline{Z}^\dagger} - K^\dagger)g_2 = 0 \\ (\partial_{\underline{Z}^\dagger} - K^\dagger)g_1 + (\partial_{\underline{Z}} + K)g_2 = 0 \end{cases} \\ &\iff \begin{cases} (\partial_{\underline{X}} + k)g_1 - (\partial_{\underline{X}} + k)g_2 = 0 \\ (\partial_{\underline{X}|} + k)g_1 + (\partial_{\underline{X}|} + k)g_2 = 0 \end{cases} \end{aligned}$$

In particular, for a matrix function of the form \mathbf{G}_0 with only one nontrivial entry g , the system reduces to

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^{\mathcal{K}}[\mathbf{G}_0] = \mathbf{0} \iff \begin{cases} (\partial_{\underline{Z}} + K)g = 0 \\ (\partial_{\underline{Z}^\dagger} - K^\dagger)g = 0 \end{cases} \iff \begin{cases} (\partial_{\underline{X}} + k)g = 0 \\ (\partial_{\underline{X}^\dagger} + k)g = 0 \end{cases}$$

Looking at this reduced system for the case of \mathbf{G}_0 , we may now also give the following definition.

Definition 4 *A continuously differentiable function g in $\Omega \subset \mathbb{R}^{2n}$ with values in \mathbb{C}_{2n} is called (left) Helmholtz Hermitian monogenic (or (left) Helmholtz h-monogenic) in Ω , iff it satisfies in Ω the system $(\partial_{\underline{Z}} + K)g = 0 = (\partial_{\underline{Z}^\dagger} - K^\dagger)g$ or, equivalently, the system $(\partial_{\underline{X}} + k)g = 0 = (\partial_{\underline{X}^\dagger} + k)g$.*

We will now establish some important integral representation formulae in the above framework. To this end, however, we need some additional notations and definitions.

For arbitrary, but fixed, $k \in \mathbb{R}$ and the corresponding constants (3) and (6), introduce the following matrix operators:

$$\begin{aligned} \mathcal{C}_\Gamma^K[\mathbf{G}_2^1](\underline{Y}) &= \int_\Gamma \mathcal{E}_K(\underline{Z} - \underline{V}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{X}) d\mathcal{H}^{2n-1}, \quad \underline{Y} \in \Omega^\pm \\ \mathcal{T}_\Omega^K[\mathbf{F}_2^1](\underline{Y}) &= - \int_\Omega \mathcal{E}_K(\underline{Z} - \underline{V}) \mathbf{F}_2^1(\underline{X}) dW(\underline{Z}, \underline{Z}^\dagger), \quad \underline{Y} \in \mathbb{R}^{2n} \\ \mathcal{H}_\Gamma^K[\mathbf{G}_2^1](\underline{U}) &= 2 \int_\Gamma \mathcal{E}_K(\underline{Z} - \underline{W}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{X}) d\mathcal{H}^{2n-1}, \quad \underline{U} \in \Gamma, \end{aligned}$$

where the Hermitian Cauchy kernel \mathcal{E}_K is given by (9), and all other notations are as introduced in Section 3.

It may then be proven that the propositions below hold. Seen the geometric assumptions on the domain, the proofs proceed along well-known lines, mimicing arguments from the Hermitian Dirac case ($k = 0$). A crucial argument however is that the Cauchy kernels \mathcal{E} and \mathcal{E}_K behave similarly, as follows from (7)–(8).

Proposition 1 *Let $\mathbf{G}_2^1 \in \mathbf{C}^1(\overline{\Omega})$. Then*

$$\mathcal{C}_\Gamma^K[\mathbf{G}_2^1](\underline{Y}) + \mathcal{T}_\Omega^K[\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1](\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

Proposition 2 *Let $\mathbf{G}_2^1 \in \mathbf{C}^1(\overline{\Omega}) \cap \mathbf{HM}^{\mathcal{K}}(\Omega)$. Then*

$$\mathcal{C}_\Gamma^K[\mathbf{G}_2^1](\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

Proposition 3 *Let $\mathbf{G}_2^1 \in \mathbf{C}^1(\Omega) \cap \mathbf{C}(\overline{\Omega})$. Then*

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^\mathcal{K} \mathcal{T}_\Omega^K[\mathbf{G}_2^1](\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

The following statement, which is a direct consequence of Proposition 2 and of the asymptotic behaviour of $\mathcal{E}_K(\underline{X})$ as $|\underline{X}| \rightarrow \infty$ (and in fact also of Remark 2), can be seen as an analogue of the complex Liouville theorem.

Proposition 4 *Let $\mathbf{G}_2^1 \in \mathbf{HM}^\mathcal{K}(\mathbb{R}^{2n})$ and*

$$\|\mathbf{G}_2^1\| = O\left(|\underline{X}|^{-\frac{2n-1}{2}}\right), \quad \text{as } |\underline{X}| \rightarrow \infty$$

Then $\mathbf{G}_2^1 = \mathbf{0}$ in \mathbb{R}^{2n} .

Next we also prove the structural analogue of the Plemelj–Sokhotski formulae.

Proposition 5 *Let $\mathbf{G}_2^1 \in \mathbf{C}^{0,\nu}(\Gamma)$. Then the boundary values of the Helmholtz Hermitian Cauchy integral $\mathcal{C}_\Gamma^K[\mathbf{G}_2^1]$ are given, in an arbitrary point $\underline{U} \in \Gamma$, by*

$$\lim_{\substack{\underline{Y} \rightarrow \underline{U} \\ \underline{Y} \in \Omega^\pm}} \mathcal{C}_\Gamma^K[\mathbf{G}_2^1](\underline{Y}) = (-1)^{\frac{n(n+1)}{2}} (2i)^n \left(\pm \frac{1}{2} \mathbf{G}_2^1(\underline{U}) + \frac{1}{2} \mathcal{H}_\Gamma^K[\mathbf{G}_2^1](\underline{U}) \right)$$

Proof.

We have

$$\begin{aligned} \lim_{\substack{\underline{Y} \rightarrow \underline{U} \\ \underline{Y} \in \Omega^\pm}} \mathcal{C}_\Gamma^K[\mathbf{G}_2^1](\underline{Y}) &= \lim_{\substack{\underline{Y} \rightarrow \underline{U} \\ \underline{Y} \in \Omega^\pm}} \int_\Gamma \mathcal{E}_K(\underline{Z} - \underline{V}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{X}) \, d\mathcal{H}^{2n-1} \\ &= \lim_{\substack{\underline{Y} \rightarrow \underline{U} \\ \underline{Y} \in \Omega^\pm}} \int_\Gamma \mathcal{E}(\underline{Z} - \underline{V}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{X}) \, d\mathcal{H}^{2n-1} \\ &\quad + \lim_{\substack{\underline{Y} \rightarrow \underline{U} \\ \underline{Y} \in \Omega^\pm}} \int_\Gamma [\mathcal{E}_K(\underline{Z} - \underline{V}) - \mathcal{E}(\underline{Z} - \underline{V})] \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{X}) \, d\mathcal{H}^{2n-1} \end{aligned}$$

The kernel of the last integral, viz

$$\mathcal{E}_K - \mathcal{E} = \begin{pmatrix} (E_k - E) + i(E|_k - E|) & -(E_k - E) + i(E|_k - E|) \\ -(E_k - E) + i(E|_k - E|) & (E_k - E) + i(E|_k - E|) \end{pmatrix}$$

behaves as $O(|\underline{X}|^{-2n+2})$ for $|\underline{X}| \rightarrow 0$, whence it is weakly singular, see e.g. [25]. Therefore this integral represents a compact operator, implying that the considered limit exists, and

$$\begin{aligned} \lim_{\substack{\underline{Y} \rightarrow \underline{U} \\ \underline{Y} \in \Omega^\pm}} \mathcal{C}_\Gamma^K[\mathbf{G}_2^1](\underline{Y}) &= \pm \frac{1}{2} \mathbf{G}_2^1(\underline{U}) + \int_\Gamma \mathcal{E}(\underline{Z} - \underline{U}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{X}) \, d\mathcal{H}^{2n-1} \\ &\quad + \int_\Gamma [\mathcal{E}_K(\underline{Z} - \underline{U}) - \mathcal{E}(\underline{Z} - \underline{U})] \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{X}) \, d\mathcal{H}^{2n-1} \end{aligned}$$

which completes the proof. \square

We now put

$$\mathcal{P}_K^\pm[\mathbf{G}_2^1](U) = \lim_{\substack{Y \rightarrow U \\ Y \in \Omega^\pm}} \mathcal{C}_\Gamma^K[\mathbf{G}_2^1](Y)$$

The operators \mathcal{P}_K^\pm are mutually complementary projection operators onto the traces of $\mathbf{HM}^\mathbf{K}(\Omega^\pm)$, respectively. In turn, this gives the necessary and sufficient conditions for the existence of a Helmholtz \mathbf{H} -monogenic extension of a given circulant matrix function in $\mathbf{C}^{0,\nu}(\Gamma)$ to Ω^+ or Ω^- , vanishing at infinity.

We immediately have the following analogue of Theorem 4.

Theorem 5 *The perturbed Hermitian Hilbert transform \mathcal{H}_Γ^K shows the following properties:*

- \mathcal{H}_Γ^K is a bounded operator on $\mathbf{C}^{0,\nu}(\Gamma)$, i.e., there exist a real constant c_Γ such that for any $\mathbf{G}_2^1 \in \mathbf{C}^{0,\nu}(\Gamma)$

$$\|\mathcal{H}_\Gamma^K[\mathbf{G}_2^1]\|_{\nu,\Gamma} \leq c_\Gamma \|\mathbf{G}_2^1\|_{\nu,\Gamma}$$

- \mathcal{H}_Γ^K is an involution on $\mathbf{C}^{0,\nu}(\Gamma)$, that is, $(\mathcal{H}_\Gamma^K)^2 = \mathbf{1}_0$.

Here c_Γ denotes a generic constant depending only on Γ .

Proof.

For the case $k = 0$ the above statement is nothing but Theorem 4. So, let $k \neq 0$. Then, since the kernel of $\mathcal{H}_\Gamma^K - \mathcal{H}_\Gamma$ is $\mathcal{E}_K - \mathcal{E}$, we have, as a result of the theory of weakly singular integral operators, see e.g. [25] or [26, Proposition 3.1], that \mathcal{H}_Γ^K coincides with \mathcal{H}_Γ up to a compact operator. Hence the statement holds. \square

Corollary 2 *The projector \mathcal{P}_K^- is a bounded operator on $\mathbf{C}^{0,\nu}(\Gamma)$, i.e., there exists a real constant c_Γ , only depending on Γ , such that for any $\mathbf{G}_2^1 \in \mathbf{C}^{0,\nu}(\Gamma)$*

$$\|\mathcal{P}_K^-[\mathbf{G}_2^1]\|_{\nu,\Gamma} \leq c_\Gamma \|\mathbf{G}_2^1\|_{\nu,\Gamma}. \quad (10)$$

Remark 3 *Observe that we can immediately reformulate the main results of this section in the framework of Lebesgue p -integrable matrix functions. For completeness, we state in the following theorem the \mathbf{L}_p boundedness of our matrix singular operators $\mathcal{H}_\Gamma^K, \mathcal{P}_K^\pm$ on a Lipschitz hypersurface.*

Theorem 6 *Let $1 < p < \infty$. The matrix singular operators $\mathcal{H}_\Gamma^K, \mathcal{P}_K^\pm$ are bounded linear operators on $\mathbf{L}_p(\Gamma)$ with*

$$\|\mathcal{H}_\Gamma^K[\mathbf{G}_2^1]\|_p \leq c_p \|\mathbf{G}_2^1\|_p,$$

$$\|\mathcal{P}_K^\pm[\mathbf{G}_2^1]\|_p \leq c_p \|\mathbf{G}_2^1\|_p,$$

for all $\mathbf{G}_2^1 \in \mathbf{L}_p(\Gamma)$ and some constant c_p depending only on p .

Finally, we do have the following characterization for the solvability of the Hermitian Helmholtz equation.

Theorem 7 *Let $\mathbf{F}_2^1 \in \mathbf{C}^1(\overline{\Omega}) \cap \mathbf{HM}^{\mathcal{K}}(\Omega)$. Then the circulant matrix function \mathbf{J}_2^1 determined by $\mathbf{J}_2^1 = \mathcal{T}^K \mathbf{G}_2^1 + \mathbf{F}_2^1$ is a solution to the matricial Helmholtz equation*

$$\Delta \mathbf{J}_2^1 + k^2 \mathbf{J}_2^1 = \mathbf{0}$$

with $k \in \mathbb{R}$, if and only if \mathbf{G}_2^1 belongs to $\mathbf{C}^1(\overline{\Omega})$ and satisfies in Ω the equation $(\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^\dagger + \mathcal{K}^\dagger)[\mathbf{G}_2^1] = \mathbf{0}$.

Proof.

It suffices to combine Proposition 3 with the factorization (4)–(5). \square

Remark 4 *Results obtained both on spaces of Hölder continuous functions, and on spaces of Lebesgue p -integrable ones constitute important building blocks for the further development of the function theory associated to the matricial Hermitian Helmholtz equation and for the treatment of the corresponding extension problems. This will be the subject of forthcoming papers.*

5 Boundary value problems of Riemann type

In view of the results established in the previous sections, it is possible to transfer the theory of Riemann boundary value problems associated to the Dirac operator (as presented in e.g. in [1, 36]) to the Hermitian matricial context. Moreover, the results of this paper may be seen as generalizations of the ones proven in [37].

We consider the Riemann boundary value problem (transmission problem) which consists in finding a circulant matrix function $\Phi_2^1 \in \mathbf{HM}^{\mathcal{K}}(\Omega^\pm)$ whose boundary values $[\Phi_2^1]^\pm$ in any point of Γ satisfy the transmission condition

$$[\Phi_2^1]^+(\underline{U}) = \mathbf{G}_2^1(\underline{U})[\Phi_2^1]^-(\underline{U}) + \mathbf{F}_2^1(\underline{U}), \quad \underline{U} \in \Gamma \quad (11)$$

and which moreover vanishes at infinity. Here \mathbf{G}_2^1 and \mathbf{F}_2^1 are given circulant matrix functions in $\mathbf{C}^{0,\nu}(\Gamma)$.

We will first treat a special Riemann boundary value problem, corresponding to a particular choice of (11).

Theorem 8 *Let $\mathbf{G}_2^1 \in \mathbf{CM}^{2 \times 2}$ be a constant matrix, i.e. independent of \underline{X} , which moreover is invertible. Then there exists a unique solution to the Riemann boundary value problem (11) and the solution may be represented by*

$$\Phi_2^1(\underline{Y}) = \mathbf{X}_2^1(\underline{Y}) \int_{\Gamma} \mathcal{E}_{\mathbf{K}}(\underline{Z} - \underline{V}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} [\mathbf{G}_2^1]^{-1} \mathbf{F}_2^1(\underline{X}) \, d\mathcal{H}^{2n-1} \quad (12)$$

where $[\mathbf{G}_2^1]^{-1}$, as usual, denotes the inverse of \mathbf{G}_2^1 , and where

$$\mathbf{X}_2^1(\underline{Y}) = \begin{cases} \mathbf{G}_2^1, & \underline{Y} \in \Omega^+ \\ \mathbf{1}_0, & \underline{Y} \in \Omega^- \end{cases}$$

Proof.

We directly see that matrix function (12) belongs to $\mathbf{HM}^\kappa(\Omega^\pm)$. Moreover, application of Proposition 5 yields

$$\begin{aligned} & [\Phi_2^1]^+(\underline{U}) - \mathbf{G}_2^1[\Phi_2^1]^-(\underline{U}) \\ &= \frac{1}{2}[\mathbf{X}_2^1]^+(\underline{U})[\mathbf{G}_2^1]^{-1}\mathbf{F}_2^1(\underline{U}) + \mathbf{G}_2^1 \int_{\Gamma} \mathcal{E}_K(\underline{Z} - \underline{U})\mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)}[\mathbf{G}_2^1]^{-1}\mathbf{F}_2^1(\underline{X}) \, d\mathcal{H}^{2n-1} \\ & \quad + \frac{1}{2}\mathbf{G}_2^1[\mathbf{X}_2^1]^-(\underline{U})[\mathbf{G}_2^1]^{-1}\mathbf{F}_2^1(\underline{U}) \\ & \quad - \mathbf{G}_2^1 \int_{\Gamma} \mathcal{E}_K(\underline{Z} - \underline{U})\mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)}[\mathbf{G}_2^1]^{-1}\mathbf{F}_2^1(\underline{X}) \, d\mathcal{H}^{2n-1} \\ &= \mathbf{F}_2^1(\underline{U}) \end{aligned}$$

whence the matrix function (12) indeed is a solution of the problem (11). The uniqueness of the solution Φ_2^1 follows by application of Proposition 4 to the auxiliary function $[\mathbf{X}_2^1]^{-1}[\Phi_2^1 - \Psi_2^1]$, where Ψ_2^1 is assumed to be another solution of the problem. This completes the proof. \square

Now we will establish an explicit reduction of the problem (11) to an equivalent singular integral equation. To that end, notice that we may assume the solution of (11) to be of the form

$$\Phi_2^1(\underline{X}) = \int_{\Gamma} \mathcal{E}_K(\underline{Z} - \underline{V})\mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)}\phi_2^1(\underline{X}) \, d\mathcal{H}^{2n-1} \quad (13)$$

where $\phi_2^1 \in \mathbf{C}^{0,\nu}(\Gamma)$ should satisfy the following singular integral equation:

$$\mathcal{P}_K^+[\phi_2^1](\underline{U}) = \mathbf{G}_2^1(\underline{U})\mathcal{P}_K^-[\phi_2^1](\underline{U}) + \mathbf{F}_2^1(\underline{U}), \quad \underline{U} \in \Gamma \quad (14)$$

Conversely, if $\phi_2^1 \in \mathbf{C}^{0,\nu}(\Gamma)$ represents a solution of (14), then the corresponding function (13) is a solution of (11). Combining these observations with Proposition 5 yields

$$\phi_2^1(\underline{U}) = [\Phi_2^1]^+(\underline{U}) - [\Phi_2^1]^-(\underline{U}), \quad \underline{U} \in \Gamma$$

whence (14) can be rewritten as

$$\phi_2^1(\underline{U}) = (\mathbf{G}_2^1(\underline{U}) - \mathbf{1}_0)\mathcal{P}_K^-[\phi_2^1](\underline{U}) + \mathbf{F}_2^1(\underline{U}), \quad \underline{U} \in \Gamma \quad (15)$$

We may now state the following result.

Theorem 9 *Let \mathbf{G}_2^1 and \mathbf{F}_2^1 be circulant matrix functions in $\mathbf{C}^{0,\nu}(\Gamma)$. Under the assumption that*

$$c_\Gamma \|\mathbf{G}_2^1 - \mathbf{1}_0\|_{\nu, \Gamma} < 1 \quad (16)$$

with c_Γ the constant appearing in (10), the Riemann boundary value problem (11) has a unique solution.

Proof.

Let us denote the integral operator appearing at the right hand side of (15) by χ_K , then we have that

$$\begin{aligned} & \|\chi_K[\phi_2^1](\underline{U}) - \chi_K[\phi_2^1](\underline{U})\|_{\nu, \Gamma} \\ &= \|(\mathbf{G}_2^1(\underline{U}) - \mathbf{1}_0) \mathcal{P}_K^-[\phi_2^1](\underline{U}) - (\mathbf{G}_2^1(\underline{U}) - \mathbf{1}_0) \mathcal{P}_K^-[\psi_2^1](\underline{U})\|_{\nu, \Gamma} \\ &\leq \|\mathbf{G}_2^1(\underline{U}) - \mathbf{1}_0\|_{\nu, \Gamma} \|\mathcal{P}_K^-[\phi_2^1 - \psi_2^1](\underline{U})\|_{\nu, \Gamma} \end{aligned}$$

for any two circulant matrix functions ϕ_2^1 and ψ_2^1 from $\mathbf{C}^{0, \nu}(\Gamma)$. On account of Corollary 2, this further yields

$$\|\chi_K[\phi_2^1](\underline{U}) - \chi_K[\psi_2^1](\underline{U})\|_{\nu, \Gamma} \leq \|\mathbf{G}_2^1(\underline{U}) - \mathbf{1}_0\|_{\nu, \Gamma} c_\Gamma \|(\phi_2^1 - \psi_2^1)(\underline{U})\|_{\nu, \Gamma}$$

or still, taking into account the assumption (16),

$$\|\chi_K[\phi_2^1](\underline{U}) - \chi_K[\psi_2^1](\underline{U})\|_{\nu, \Gamma} < \|(\phi_2^1 - \psi_2^1)(\underline{U})\|_{\nu, \Gamma}$$

This implies that the integral operator χ_K on $\mathbf{C}^{0, \nu}(\Gamma)$ satisfies the contractive mapping principle, see e.g. [30], whence there exists a unique solution of (14) and thus also a unique solution of (11).

Remark 5 *We can extend the scope of our results obtained on Hölder spaces to the much larger class of Lebesgue p -integrable matrix functions in order to solve the Riemann boundary value problem (11) with \mathbf{L}_p data. The \mathbf{L}_p boundedness of the involved operators, as stated in Theorem 6, leads to solvability results on those classes of functions, which are direct translation of the corresponding Theorems 8–9 to the \mathbf{L}_p setting. Again, all formulas then have to be reinterpreted as mentioned in Remark 1.*

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